Notes VI: Review and Exercises; Transport in Random Media

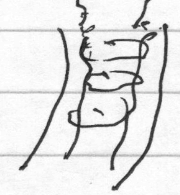
Review

⇒ Have focused on computing $\chi_{\perp\, eff}$ for stochastic fields, as representative problem of transport. Have ignoring feedback, though see exercises.

⇒ can identify:

- $Ku < 1$ ⇒ $D_M \sim \langle \tilde{b}^2 \rangle l_{ac}$ ⇒ $\chi_\perp$
  ↳ (parallel dynamics)

  diffusive, 3D problem
  ⟺ resonances critical



- $Ku > 1$ ⇒ $D_M \sim \langle \tilde{b}^2 \rangle^{1/2} \Delta_\perp$ ⇒ $\chi_\perp$
  (3rd dim?)

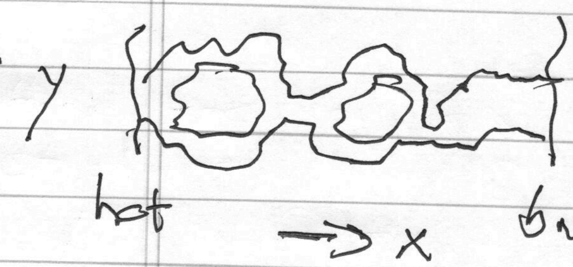
  percolative, 2D problem
  ⟺ everywhere in resonance

recall: $\frac{d\underline{x}}{dz} = \nabla \tilde{A} \times \hat{z}$, in high Ku regime
(stochastic)

⇒ lines: $\nabla \tilde{A} \cdot d\underline{x} = 0$

⇒ Problem is one of "stochastic topography" ⇒ trace paths on 2D, z akin time.

i.e.



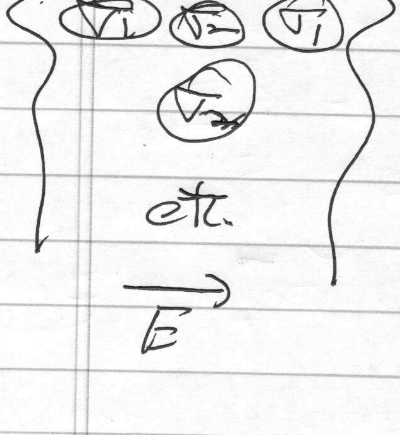
n.b.: do extended lines, exist? What of $x_{+,0}$ → jump from cell to cell?

seek contours that "connect" hot to cold boundaries.

⇒ For high Ku, problem effectively one of effective transport coefficient in 2D random medium, ⇒ classic is conductivity.

So, examine transport, especially effective medium theory, for 2D random media

i.e. Dykhne problem → effective conductivity



QLT: $\tilde{\sigma}/\bar{\sigma} \ll 1$

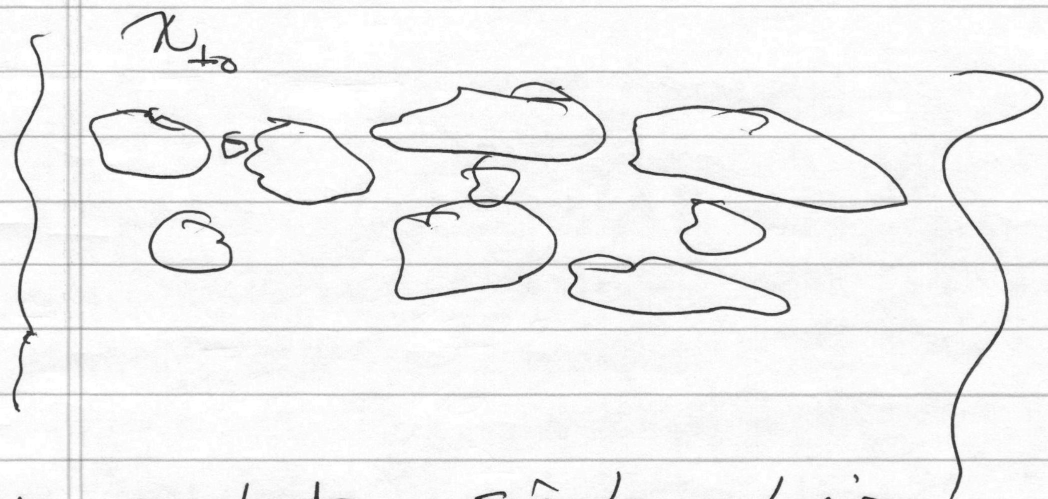
$$\sigma_{eff} = \bar{\sigma} - \frac{\langle \tilde{\sigma}^2 \rangle}{2\bar{\sigma}}$$

For equal # domains of σ_1, σ_2:

$$\boxed{\sigma_{eff} = (\sigma_1 \sigma_2)^{1/2} \text{; exact.}}$$

Note that if pure 2D problem $\nabla \cdot \underline{b} = 0$ ensures all field lines in domain <u>closed</u>.

i.e.



- here exclude single line spanning wall-to-wall, though possible in reality, i.e. edge layer.

- back to theme of $\chi_{\perp 0}$ (ambient, collisional, non-magnetic transport) scattering from closed loop to closed loop <u>or</u> introducing weak three dimensionality

- will expect $\chi_{\perp eff} \sim \chi_{\perp 0}^{\alpha} (\tilde{b}^2 \chi_{\parallel})^{\gamma}$, hybrid

- points toward cell + layer problem.

n.b.: $\bar{\sigma} = (\sigma_1 + \sigma_2)/2$

$\tilde{\sigma} = (\sigma_1 - \sigma_2)/2$

$\Rightarrow$

$\sigma_1 = \bar{\sigma} + \tilde{\sigma}$

$\sigma_2 = \bar{\sigma} - \tilde{\sigma}$

$$\sigma_1 \sigma_2 = \left(\bar{\sigma}^2 - \tilde{\sigma}^2\right)^{1/2}$$

$\tilde{\sigma}/\bar{\sigma} \ll 1$ $$\cong \bar{\sigma}\left(1 - \frac{1}{2}\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\right) \checkmark$$

QL and Dykhne correct in small fluctuation limit. ✓ But interesting case is when $\tilde{\sigma}$ <u>not</u> small, i.e. $\sigma_1 \gg \sigma_2$

$$\sigma_{eff} \sim \left[\sigma_1 \left(\sigma_2/\sigma_1\right)\right]^{1/2} \sim \sigma_1^{1/2} \left(\sigma_2/\sigma_1\right)^{1/2}$$

$\Rightarrow$ Focus on:

- random, effective medium theories, especially 2D
- percolation theory.

Exercises

i.) [redacted] Compute τ_{eff} for 2D, 3D using a quasilinear closure, taking $\tilde{\tau} \ll \tau$.

How does the result compare to the exact one?

ii.) Re-visit the $Ku < 1$, hydrodynamic result of Kadomtsev and Pogutse this time assuming a spectrum of magnetic fluctuations with finite correlation time τ_{c_k}, i.e.

$$\langle \tilde{b}^2 \rangle_{k} = \langle \tilde{b}(0)\tilde{b}(\tau) \rangle_{k} = \langle \tilde{b}^2 \rangle_{k} e^{-|\tau|/\tau_{c_k}}.$$

How does the prediction for $\chi_{\perp eff}$ change?

iii.) Consider the hydrodynamic calculation of $\chi_{\perp eff}$ in a sheared cylinder.

a.) Derive the effective $\chi_\perp$

b.) Derive the effective $\Delta_{\perp c_k}$.

c.) When would it be necessary to to re-consider the Kubo number here?

iv.) Consider flow of form:

$$\underline{V} = V_0 \cos ky \cos \omega t \, \hat{x}$$

Assume molecular diffusion D_0, so, show:

$$D_{eff} = D_0 + \frac{1}{2} k^2 V_0^2 D_0 \Big/ \omega^2 + (k^2 D_0)^2$$

$$= D_0 \left(1 + \frac{k^2 V_0^2}{2} \Big/ \left(\omega^2 + k^2 D_0{}^2 \right) \right)$$

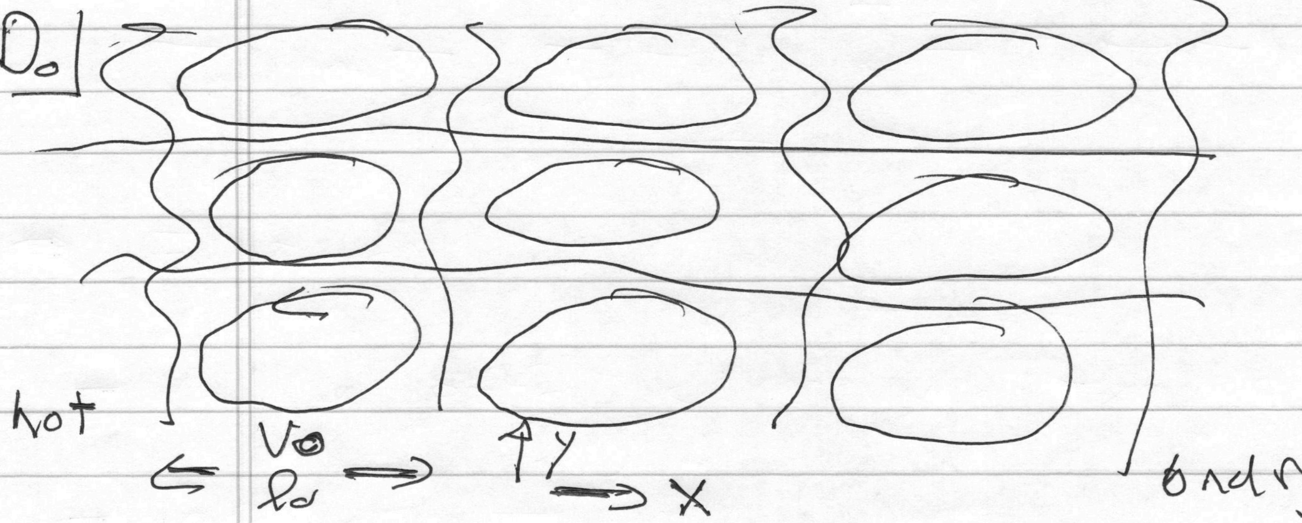
Transport in Random Media

→ Have argued that transport in high Ku regime is similar to transport in random media

→ Have discussed Dykhne calculation for effective conductivity of mixture of elements σ_1, σ_2: $\sigma_{eff} = (\sigma_1 \sigma_2)^{1/2}$.

Now consider related Taylor problem, relevant to system close to marginality.

Point is medium of neighboring non-overlapping convective cells, i.e.



medium has: molecular diffusivity D_0

cells: scale l_0, speed V_0

For passive scalar problem:

$$\frac{\partial n}{\partial t} + \underline{v}\cdot\underline{\nabla} n - D_0 \nabla^2 n = 0$$

can define: $P_0 = V_0 l_0 / D_0$
→ Peclet number
→ $P_0 \gg 1$ of interest

Interest: – Effect transport coefficient, i.e. diffusivity, for scales $L \gg l_0$

(*) – effective medium problem with 2 transport process:

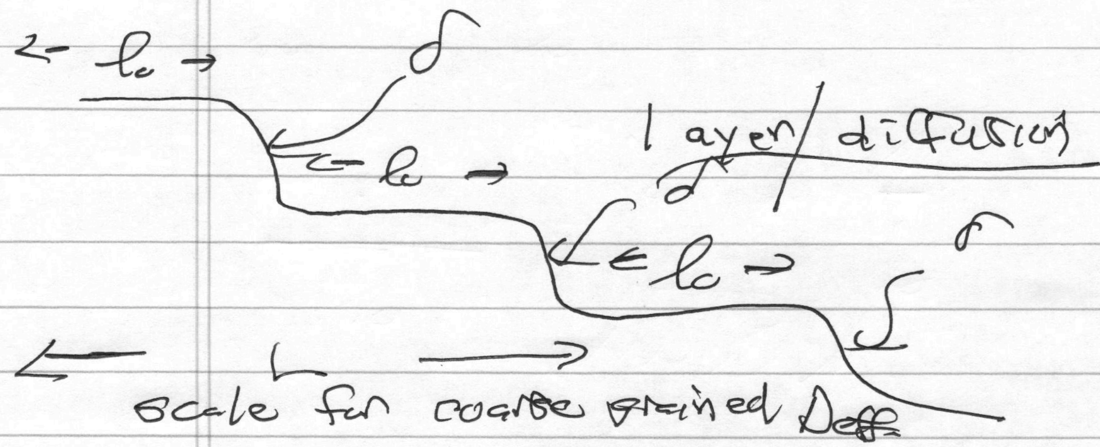
fast → convection → operating in cells
slow → diffusion → operating in boundary layers.

What is D_{eff}?

Point: – transport is hybrid of
- fast kicks thru cell
- slow diffn thur BL

– diffusion is ultimate origin of irreversibility for static cells. Only BL particles transported.

Can envision concentration profile:



Heuristic argument:

→ for random walk:

$$D_{eff} \simeq \text{factor} \, \frac{(\Delta x)^2}{\Delta t}$$

factor: active fraction for diffusion

$\sim \delta / l_0$ small in BL thickness

Δt: cell circulation time

$\sim l_0 / V_0$

then

$$\delta^2 \sim D_0 \Delta t \sim D_0 \frac{l_0}{V_0}$$

and $\Delta X \sim l_0$ (cell scale).

so

$$D_{eff} \cong \frac{\delta}{l_0} \frac{l_0^2}{l_0/V_0}$$

$$\cong \left(D_0 \frac{l_0}{V_0}\right)^{1/2} \frac{\cancel{l_0} V_0}{\cancel{l_0}}$$

$$\cong \left(D_0 V_0 l_0\right)^{1/2}$$

$$\boxed{D_{eff} \cong \left(D_0 D_{cell}\right)^{1/2}} \cong D_0 \left(P_e\right)^{1/2}$$

→ D_{eff} is geometric mean of D_0 (slow) and D_{cell} (fast), $\sim P_e^{1/2}$, large P_e

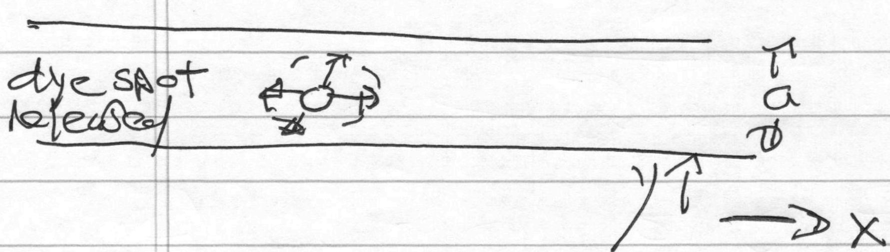
→ resembles Dykhne result, but
⇒ Dykhne → equal areas σ_1, σ_2
cells → $\delta_{active}/l_0 \ll 1$

→ see Rosenbluth, et. al. 1987 for details of calculation (tedious).

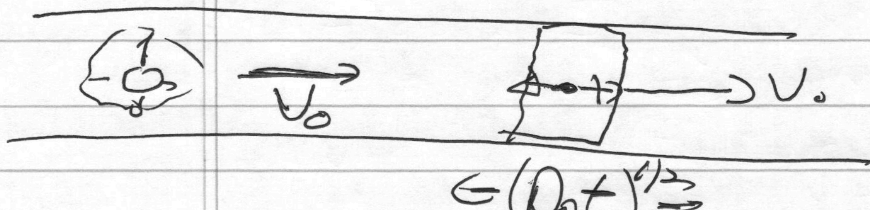
Related problem: <u>Shear dispersion</u>

see Taylor, 1953 et seq. (many postings)

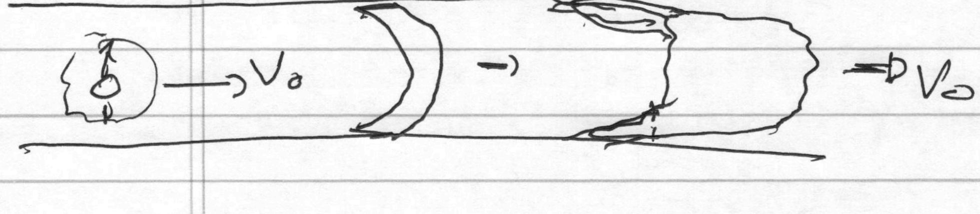
Problem stated by comparison of <u>three</u> laminar flows, into which dye with molecular diffusion D_0 is injected.



Flow ①
stationary water.



Pipe ②
Flow at V_0.
Slip BC



Pipe ③
No-slip BC

Pipe ② : - Slug CM advects at V_0

- slug expands axially at $(D_0 t)^{1/2}$ ⇒ molecular diffusion

⇒ Fact, Pipe ③:

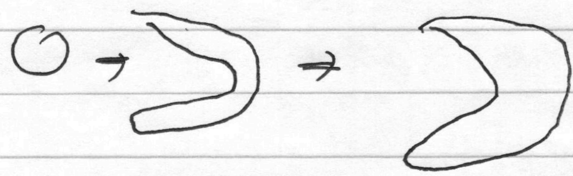
- experiments (cf 1953 paper) indicate more rapid dispersal of dye in sheared flow, i.e. effective axial diffusion enhanced

$$D_{eff\ axial} > D_0$$

$$\Rightarrow \quad D_{eff} = D_0 + D_{shear}$$

- though CM velocity same, U_0

- velocity shear stretches cloud,



spreading it more rapidly. Effect is shear + D_0

How calculate?

Consider dye concentration field $c(x, y, t)$

Then,

$$\underline{\nabla} \cdot \underline{v} = 0$$

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla} c = D_0 \nabla^2 c$$

and consider: $c = \langle c \rangle + \tilde{c}$

mean concentration

$$\langle c \rangle = \langle c(x,y) \rangle_y = \int_{-a/2}^{a/2} dy\, c(x,y)$$

$\rightarrow \langle c(x) \rangle$ — y int.

Likewise: $\underline{v} = V(y)\,\hat{x} + \tilde{\underline{v}}$

$$\frac{\partial}{\partial t}\langle c \rangle + \frac{\partial}{\partial t}\tilde{c} + \langle v \rangle \cdot \nabla \langle c \rangle + \tilde{\underline{v}} \cdot \nabla \langle c \rangle + \langle v \rangle \cdot \nabla \tilde{c} + \nabla \cdot \langle \tilde{v} \tilde{c} \rangle - D_0 \nabla^2 \langle c \rangle - D_0 \nabla^2 \tilde{c} = 0$$

here: $\langle v \rangle = \int_{-a/2}^{a/2} dy\, V(y)$ cross stream avg.

Seek: $\nabla \cdot \langle \tilde{V} \tilde{C} \rangle = \frac{\partial}{\partial x} \langle \tilde{V}_x \tilde{C} \rangle$

Note: $\tilde{V}_x = \underset{\text{exact}}{V(y)} - \underset{\text{mean}}{\langle V \rangle}$ $\Rightarrow$ deterministic

Need $\tilde{C}$:

$$\partial_t \tilde{C} - D_0 \nabla^2 \tilde{C} = -\tilde{V}_x \frac{\partial}{\partial x} \langle C \rangle$$

take out transformation to $\bar{V}$ frame.

Stationary problem:

$$-D_0 \nabla^2 \tilde{C} = -\tilde{V}_x \frac{\partial}{\partial x} \langle C \rangle$$

$$D_0 k_y^2 \tilde{C}_{k_y} = -\tilde{V}_{x\,k_y} \frac{\partial}{\partial x} \langle C \rangle$$

$$\boxed{k_y^2 > (2\pi)^2 / a^2} = -\tilde{V}_{x\,k_y} \frac{\partial}{\partial x} \langle C \rangle$$

Channel width sets min k_y

then,

$$\tilde{c}_{k_y} = -\frac{\tilde{v}_{x\,k_y}}{k_y^2 D_0}\frac{\partial \langle c \rangle}{\partial x}$$

$$\langle \tilde{v}_x \tilde{c} \rangle = -\sum_{k_y} \frac{|\tilde{v}_{k_y}|^2}{k_y^2 D_0} \frac{\partial \langle c \rangle}{\partial x}$$

$$= -D_{Flow} \frac{\partial \langle c \rangle}{\partial x}$$

$\Rightarrow$

$$\boxed{D = D_0 + \sum_{k_y} \frac{|\tilde{v}_{k_y}|^2}{k_y^2 D_0}}$$

as $\tilde{v}$ tied to mean flow
(# + bd (calcn criteria)).

$$D = D_0 + \alpha \frac{V_0^2 \, \cancel{Q}^2}{D_0}$$

so

$$D_{eff} = D_0 + \alpha \frac{v_0^2 a^2}{D_0}$$

effective diffusivity for axial spreading

N.B.:– Origin of irreversibility is D_0

- Akin to cell problem, laminar flow + molecular diffusion yield transport

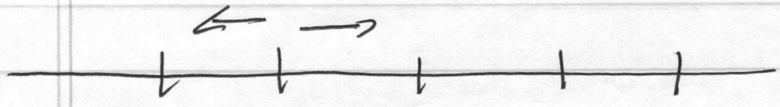
- Can generalize to pipe with turbulence → scaling ?

- Taylor originally proposed shear dispersion as mechanism to distribute nutrients, etc. in blood flow.

Percolation processes 1

Percolation vs. Diffusion

a.) Diffusion $\Rightarrow$ 1D random walk



- medium fixed
- particle motion stochastic

$$\delta x^2 \sim Dt$$

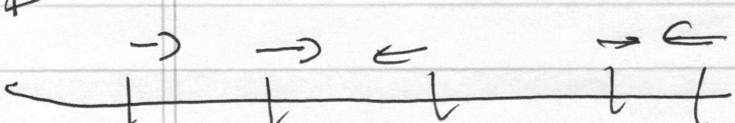
$$\sim N$$

$\hookrightarrow$ # steps

and particle returns.

b.) Percolation - assign left/right orientation to each site, with probability 1/2

particle $\Rightarrow$



etc.

- medium stochastic
- particle motion deterministic.

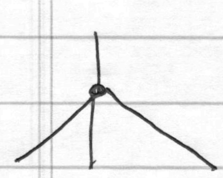
Simply: → diffusion: medium deterministic, motion stochastic

→ percolation: motion deterministic, medium stochastic

Examples:

a.) Cascade process

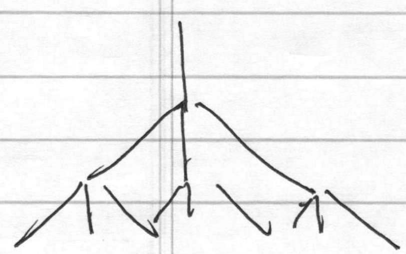
$(p+q) = 1$



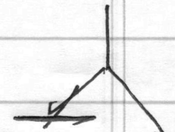
0 desc. 1 desc 2 desc

q^2 $2pq$ p^2

can think of as diffusion

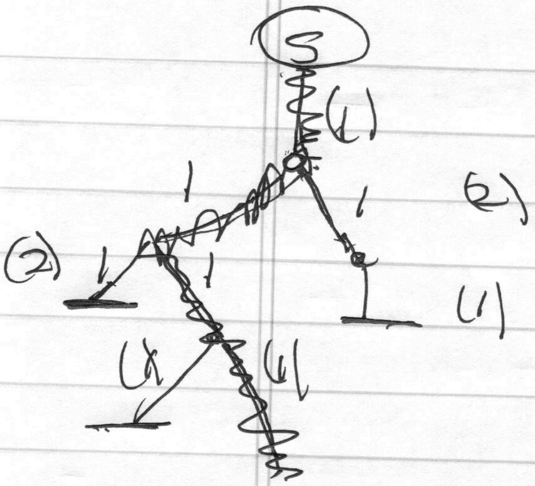


<u>or</u>, does one generation "reach" N into future



<u>so</u>

0 1 or 2



q probability of blocked
p of 1, 2 conf.

percolation

In percolation:

→ intrinsic and random properties of medium determine motion.

→ in diffn / stoch. motion indeterminate. Prop. particle fundamental (i.e. stochastic orbits).

Origin of random characteristics of medium:

→ randomly dam / cut connections
→ dimensionality matters!

→ result is random maze.

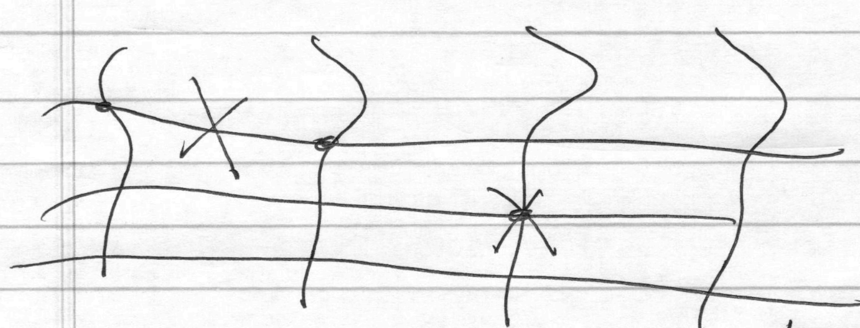
→ flow can traverse A→B only if there is an un-dammed, un-cut self-avoiding random walk connecting.

SAW visits intermediate at most once.

⇒ General aspects (mostly topological)

– can have "bond" or "site" percolation

i.e



LN

bond → some fraction of ~~XXXX~~ bonds/connections missing

site → some fraction of sites missing

⇒ game played either way.

– Consider:

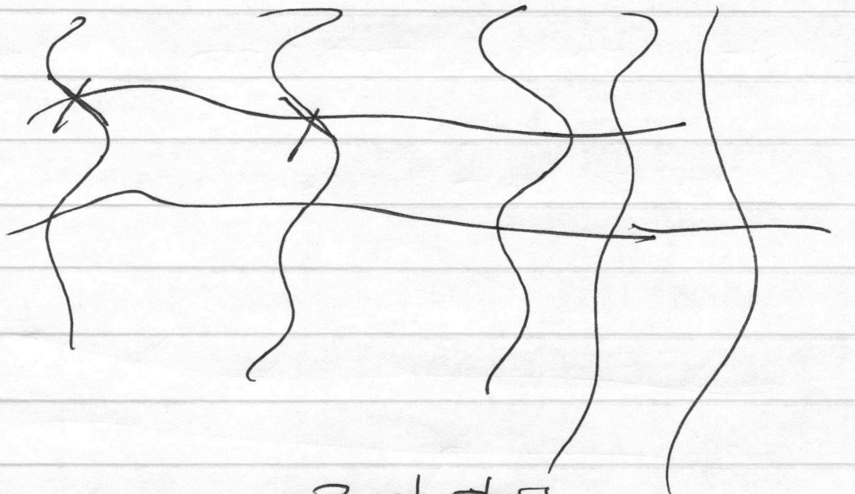
→ lattice of N sites, $N \gg 1$

→ concentration of allowed sites X

→ can envision increasing X from below, i.e.

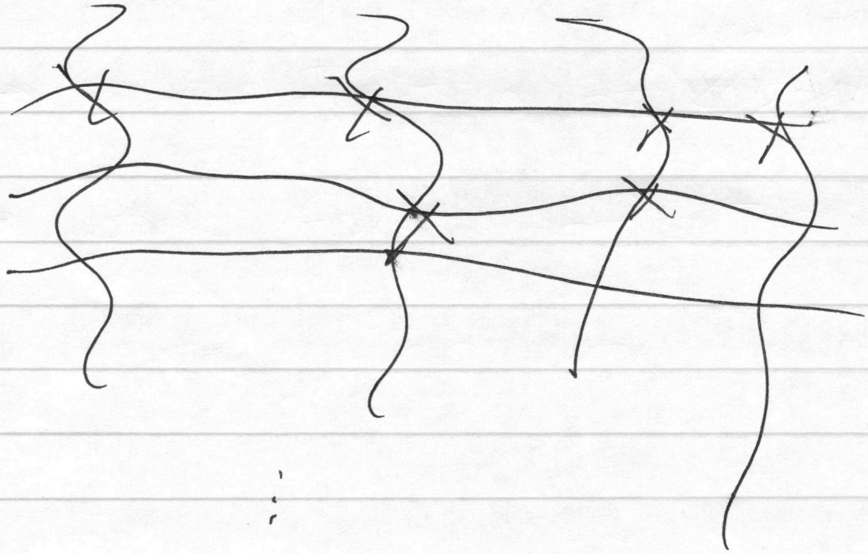
2 cluster

i.e.



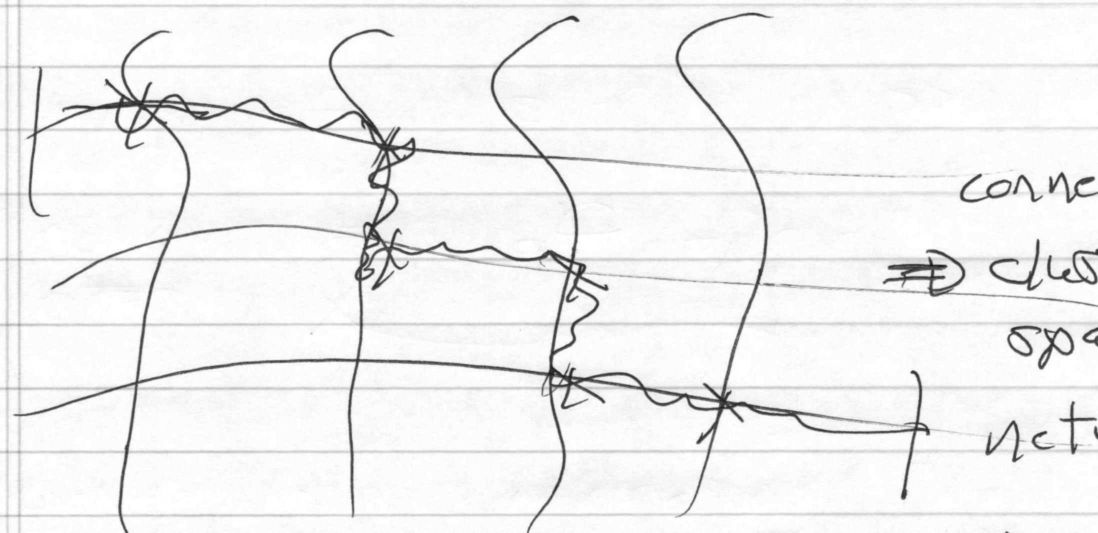
X_1

3 clusters



$X_2 > X_1$

⋮



connecting

$\Rightarrow$ cluster spans

network

can traverse.

∞ $N \to \infty$.

- $x \ll x_c \to$ isolated small clusters
- $x \uparrow \to$ larger clusters form.
- $\ell(x) \uparrow$ with x

  ℓ = size

- as $x \uparrow$, few larger clusters form
- $\ell(x_c) \to \infty$ as $N \to \infty$
- one cluster for $x > x_c$.

$P^s(x) \equiv$ site percolation probability

$\equiv$ ratio of # sites in big (infinite) cluster to # sites in lattice

$\equiv$ fraction of system in which DC conduction is possible.

And:

- there is a threshold concentration of active sites, x_c

- near threshold:

$$\rho^{\sigma}(x) \sim (x - x_c)^{s}$$

s → perc. exponent

ρ^{σ} → fraction DC conduction

- 3D: $s \sim .3 \rightarrow .4$ corresponding lattice structure

⇒

- percolation is a type of phase transition

i.e., $s \leftrightarrow$ critical exponent

$x_c \leftrightarrow$ critical temp / pt.

- key question is effective medium model near x_c.

- connection to turbulence: intermittency → reduced packing fraction.